The Degenerate Fermi Gas

Now let's study Fermions in the degenerate high-density limit $n\lambda^D \gg 1$.

Examples:

White-dwarf / metals (Copper, Fe, Al etc.)
↳ Gas of electrons liberated from ions.

$$\lambda_e = \sqrt{2\pi\hbar^2/m_e k_B T} = 74\ \text{nm}/\sqrt{T/K^\circ}$$

When inter-electron distance < λ, degenerate.

At $T = 293K^\circ$, $\lambda_e = 4\,\text{nm}$. In typical crystals, inter-atom spacing is $a \sim 0.5\,\text{nm}$, with $-e$ per ion, so certainly degenerate!

White-dwarfs are $T \sim 10^5 K^\circ \gg 293K^\circ$, but also much denser: $10^9\ \text{kg}/\text{m}^3$

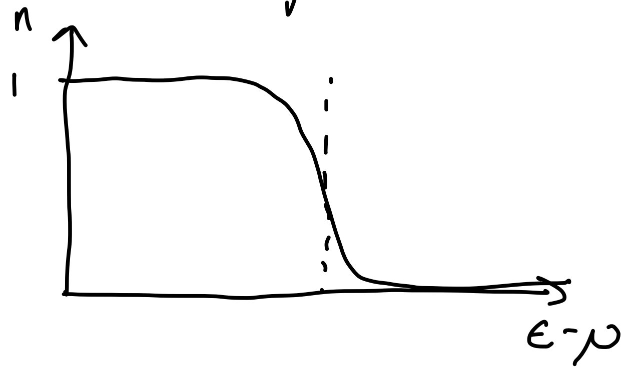
Neutron-stars: electrons and protons fuse into neutrons:

$$e_- + p \longrightarrow n_0 + \nu$$

n_0 is also fermion. $m_n \sim 1000\, m_e$ and $T \sim 10^6 K$, but insanely dense: $\rho \sim 10^{17}\,\text{kg}/\text{m}^3$

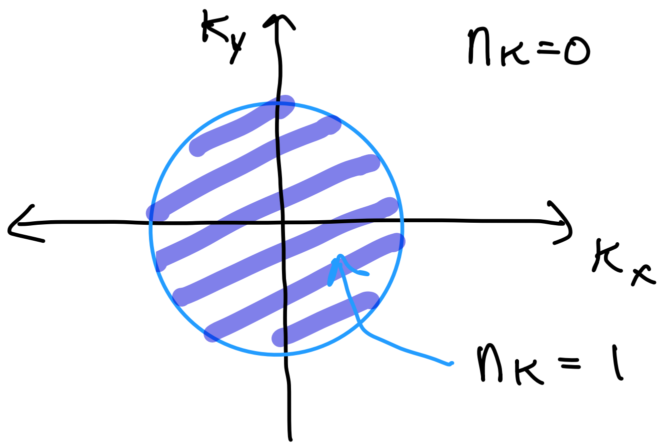
(Interestingly, in neutron-star interactions may be strong enough to cause deviations in gas behaviour ⟶ superconductivity. As in Al!)

Energetically, fermions want to pile into minima of ϵ_k @ $k=0$, but are prevented by Pauli-exclusion: $n_k \leq 1$



$$n = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

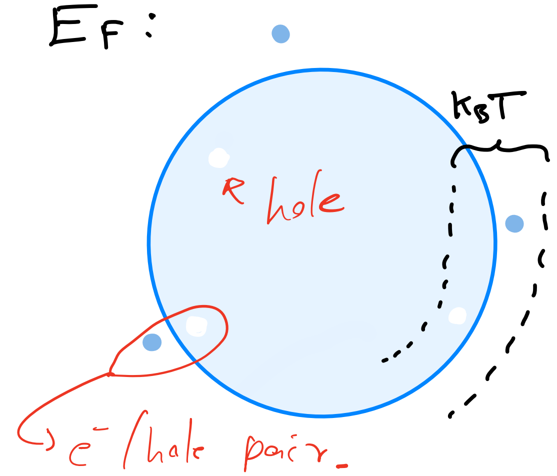
For a fixed $N = \sum_k n_k$, electrons are forced to occupy a <u>volume</u> of lowest ϵ_k states, $\epsilon_k = \frac{\hbar^2 k^2}{2m}$. At $T=0$:



The occupied volume is called the "Fermi-Sea". It's boundary is an equal-$\epsilon_k = E_F = \mu$ contour called the "Fermi-surface"

The existence of a F.S. has radical thermodynamic consequences for heat capacity, pressure, etc.

The excited states look like either the addition of $\cdot e$ with $E > E_F$ or the removal of $-e$ with $E < E_F$: a "hole". Will occur within a strip $\Delta E \sim k_B T$ of E_F:



The number of such states scales as

$$\Omega_T \sim \underset{\text{surface of D-ball}}{S^{D-1}} K_F^{D-1} \underbrace{\frac{\Delta E}{\partial E / \partial k}}_{\text{width of strip}} \propto k_B T$$

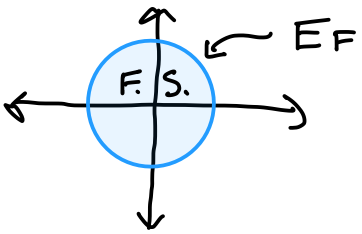
This $\Omega_T \sim k_B T$ implies state is gapless and will lead to large heat capacity ($C \propto T$) at low-T.

(compare to $\Omega_T \sim T^3$ in case of phonons)

Let's first compute the pressure at $T=0$. Classically, $P = \frac{N}{V} k_B T \to 0$. But not for fermions! We know

$$P(N,V,T) = -\frac{\partial F}{\partial V} = -\frac{\partial E}{\partial V} + T\frac{\partial S}{\partial T}.$$

At $T=0$, just need $P = -\frac{\partial E}{\partial V}$



First, what is relation between F.S @ E_F and N?

$$N = \sum_k \frac{1}{e^{\beta \epsilon_k}+1} \overset{T\to 0}{=} \sum_{k \in F.S.}$$

With $k_i \in \frac{2\pi}{L}\cdot m$, $m = \cdots, -2, -1, 0, 1, \cdots$

$$\sum_k \sim \underbrace{L^D}_{V} \int \frac{d^D k}{(2\pi)^D}$$

$$\frac{N}{V}(T=0) = \int \frac{d^D k}{(2\pi)^D}\, \Theta(E_F - \epsilon_k)$$

At this point, need to know ϵ_k, e.g.

relatavistic: $\epsilon_k = \sqrt{(mc^2)^2 + p^2c^2}$ $\quad p = \hbar k$

non-rel: $\epsilon_k = \frac{p^2}{2m}$

Crystal: $\epsilon_k = -2t\cos(k\cdot a)$ lattice spacing

Cu, "Band structure"

We'll do $\epsilon_K = \dfrac{\hbar^2 k^2}{2m}$ here. Defining

$$E_F = \frac{\hbar^2 k_F^2}{2m}, \quad \int \frac{d^D K}{(2\pi)^D} \Theta(E_F - \epsilon_K) = \int_{k<K_F} \frac{d^D K}{(2\pi)^D}$$

$$K_F = \sqrt{2m E_F/\hbar^2}$$

We first define

$$\int_{|K|<K_F} \frac{d^D K}{(2\pi)^D} K^n = \frac{S^{D-1}}{(2\pi)^D} \frac{K_F^{n+D}}{n+D} \equiv \gamma_n \cdot K_F^{n+D}$$

$$\frac{N}{V} = \gamma_0 K_F^D \qquad K_F = \left[\gamma_0^{-1} \frac{N}{V}\right]^{1/D}$$

Now $E = \sum_{K<K_F} \epsilon_K$

$$E = V \cdot \int_{K<K_F} \frac{d^D K}{(2\pi)^D} \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \gamma_2 K_F^{2+D} \cdot V$$

$$= \frac{\hbar^2}{2m} \gamma_2 \left(\gamma_0^{-1} N\right)^{1+2/D} V^{-2/D}$$

$$P = -\frac{\partial E}{\partial V} = \frac{\hbar^2}{2m} \gamma_2 \left(\gamma_0^{-1} \frac{N}{V}\right)^{1+2/D} \frac{2}{D}$$

$$= \frac{N}{V} E_F \cdot \underbrace{\frac{\gamma_2}{\gamma_0} \frac{2}{D}}_{\frac{D}{2+D} \frac{2}{D}} = \boxed{\frac{N}{V} \cdot E_F \frac{2}{D+2}}$$

This should be compared with

$$P = \frac{N}{V} k_B T \quad \text{vs} \quad P_F = \#\cdot \frac{N}{V} E_F$$

as $k_B T \to 0$.

Fermions have massively larger pressure at low-T: explains much of the hardness of solids, absence of core collapse in White Dwarf, etc.

What about $T>0$? We'd like to obtain the leading order behavior in T.

First, recap:

$$Q = \prod_k \left(1 + z e^{-\beta\epsilon_k}\right), \quad z = e^{\beta\mu}$$

$$\frac{\Omega}{V} = -\frac{1}{\beta V}\ln(Q) = \frac{-1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln\left(1+z e^{-\beta\epsilon_k}\right)$$

$$\frac{N}{V} = \int\frac{d^3k}{(2\pi)^3}\, n_F(\epsilon_k-\mu) = \int dE\, \rho(E)\, n_F(E-\mu)$$

Fermi funct.

$$\frac{E}{V} = \int\frac{d^3k}{(2\pi)^3}\, E\, n_F(\epsilon_k-\mu) = \int dE\, \rho(E)\cdot E\, n_F(E-\mu)$$

where the density $\rho(E)$ of states with energy E is given by

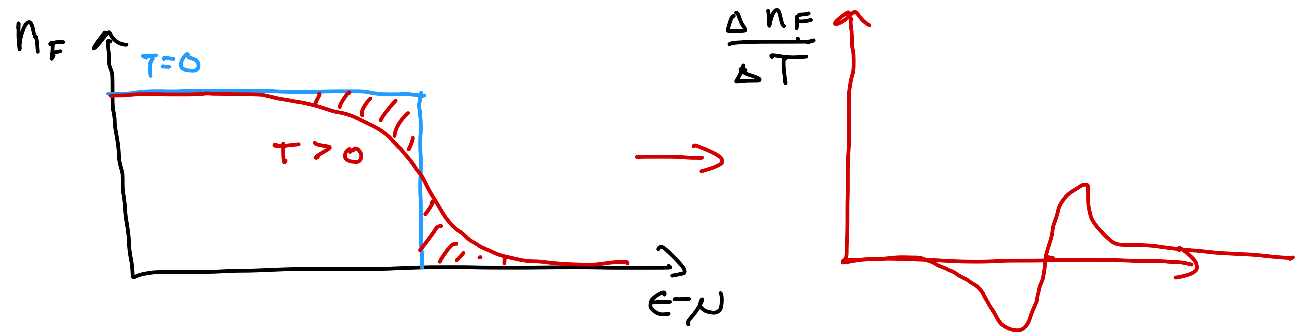
$$\rho(E) = \int\frac{d^3k}{(2\pi)^3}\,\delta(E-\epsilon_k) = V^{-1}\left.\frac{\partial N}{\partial E}\right|_{T=0}.$$

Using $\rho(E)$, we can convert any integral

$$\int\frac{d^3k}{(2\pi)^3}\, g(\epsilon_k) \longrightarrow \int dE\, \rho(E)\, g(E),$$

such that we no longer have to care about ϵ_k!

When $T \neq 0$, n_F is no longer a step function:



We would like to find rule

$$\int d\epsilon\, f(\epsilon)\, n_F(\epsilon-\mu) = f_0 + \beta^{-1} f_1 + \beta^{-2} f_2 + \cdots$$

This is called "Sommerfeld Expansion"

$$f_0 = \int_{-\infty}^{\mu} d\epsilon\, f(\epsilon), \quad f_1 = 0, \quad f_2 = \frac{\pi^2}{6} \left.\frac{\partial f}{\partial \epsilon}\right|_{\mu}$$

This is very cool: f_2 depends <u>only</u> on $f'(\mu)$, not $f(\epsilon<\mu)$! This is because as $T \to 0$, $\frac{\Delta n_F}{\Delta T}$ starts looking like an approximation of a derivative at μ!

With this in hand,

$$\frac{N}{V} = \underbrace{\int_{-\infty}^{\mu} \rho(E)}_{\frac{N}{V}(\mu, T=0)} + \frac{\pi^2}{6}\beta^{-2}\rho'(\mu)$$

$$\frac{E}{V} = \int_{-\infty}^{\mu} E\rho(E) + \frac{\pi^2}{6}\beta^{-2}\partial_E(E\rho(E))\big|_\mu$$

$$= \frac{E}{V}(\mu, T=0) + \frac{\pi^2}{6}\beta^{-2}\left(\rho(\mu) + \mu\rho'(\mu)\right)$$

Take away: at low-T, thermo depends only on $\rho(\mu)$, $\rho'(\mu)$, a property near Fermi surface!

Let's use this to compute $C_{V,N} = \frac{\partial E}{\partial T}\big|_{V,N}$

Problem is we have $N(\mu, T, V)$ $E(\mu, T, V)$

To convert,

$$E(\mu, T, V) = E(N(\mu, T, V), T, V)$$

$$\frac{\partial E}{\partial T}\Big|_\mu = \frac{\partial E}{\partial N}\Big|_T \frac{\partial N}{\partial T}\Big|_\mu + \frac{\partial E}{\partial T}\Big|_N$$

$$\frac{\partial E}{\partial \mu}\Big|_T = \frac{\partial E}{\partial N}\Big|_T \frac{\partial N}{\partial \mu}\Big|_T$$

$$C_{V,N} = \frac{\partial E}{\partial T}\Big|_N = \frac{\partial E}{\partial T}\Big|_\mu - \frac{\partial E}{\partial \mu}\Big|_T \left(\frac{\partial N}{\partial \mu}\Big|_T\right)^{-1} \frac{\partial N}{\partial T}\Big|_\mu$$

$$C_{V,N} = \underset{(A)}{\left.\frac{\partial E}{\partial T}\right|_{\mu}} - \underset{(B)}{\left.\frac{\partial E}{\partial \mu}\right|_T} \underset{(C)}{\left(\left.\frac{\partial N}{\partial \mu}\right|_T\right)^{-1}} \underset{(D)}{\left.\frac{\partial N}{\partial T}\right|_{\mu}}$$

So we need to compute 4* things

$$\frac{N}{V} = \int_{-\infty}^{\mu} \rho(E) + \frac{\pi^2}{6}\beta^{-2}\rho'(\mu)$$

So (D) $$\frac{\partial N}{\partial T} = V\frac{\pi^2}{3} k_B \beta^{-1} \rho'(\mu)$$

To leading order (β^{-1}) this means only need (B), (C) to $\mathcal{O}(1)$:

(C) $$\frac{\partial N}{\partial \mu} = V\rho(\mu) + \mathcal{O}(\beta^{-2})$$ [T=0 compressibility]

$$\frac{E}{V} = \int_{-\infty}^{\mu} E\,\rho(E) + \frac{\pi^2}{6}\beta^{-2}\left(\rho(\mu) + \mu\rho'(\mu)\right)$$

(A) $$\left.\frac{\partial E}{\partial T}\right|_{\mu} = \frac{\pi^2}{3} k_B \cdot \beta^{-1} \left(\rho(\mu) + \mu\rho'(\mu)\right)$$

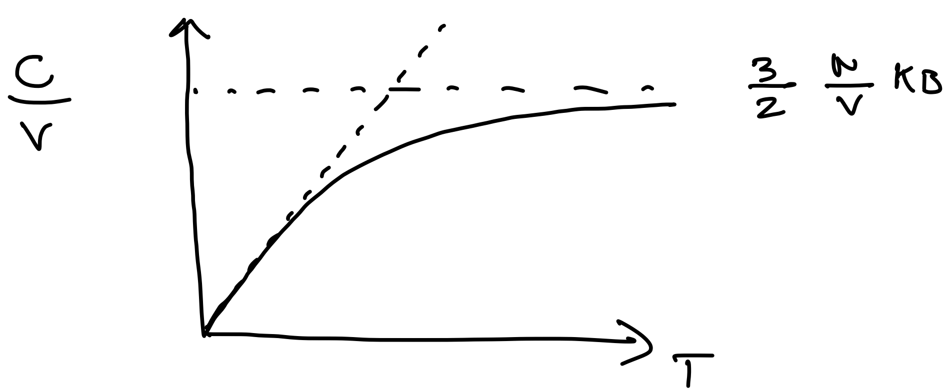
(B) $$\left.\frac{\partial E}{\partial \mu}\right|_T = V \mu\rho(\mu) + \mathcal{O}(\beta^{-2})$$

$$\left.\frac{\partial E}{\partial N}\right|_T = \left.\frac{\partial E}{\partial \mu}\right|_T \left(\left.\frac{\partial N}{\partial \mu}\right|_T\right) = \mu + \mathcal{O}(\beta^{-2})$$

$$\frac{C}{V} = \frac{\pi^2}{3} k_B \cdot \beta^{-1}\left(\rho(\mu) + \mu \rho'(\mu)\right) - \frac{\pi^2}{3} k_B \beta^{-1} \mu \rho'(\mu)$$

$$\boxed{C = V \frac{\pi^2}{3} k_B^2 \, T \cdot \rho(E_F)}$$

One of <u>the</u> most important results in solid state: the T- linear heat capacity depends only on D.O.S $\rho(E_F)$ at Fermi-surface! True regardless of ϵ_k and $\rho'(E_F)$!



Compared to gapped state $(e^{-\Delta/k_B T})$ or phonons (bosons) $(C \sim T^3)$, this is huge heat capacity. It's because the effective # of D.O.F not frozen out is $\dfrac{C}{\frac{1}{2} k_B} = \# \cdot V \cdot \rho(E_F) \cdot k_B T$