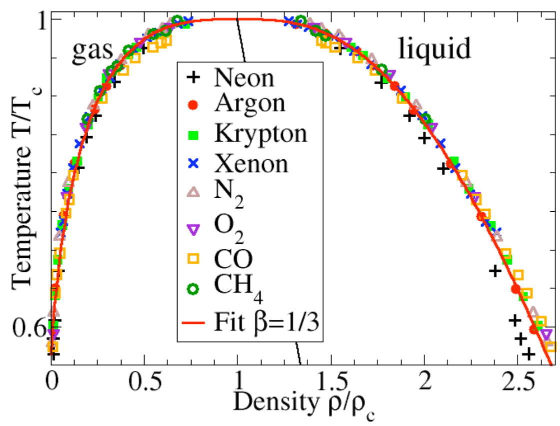
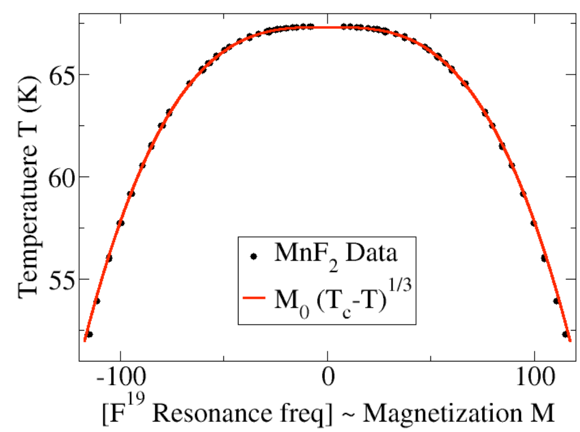
Universality: Shared Critical Behavior

Ising Model and Liquid-Gas Critical Point



Same critical exponent β=0.332!



Liquid-Gas Critical Point

$\rho-\rho_c \sim (T_c-T)^\beta$

$\rho^{Ar}(T) = A\,\rho^{CO}(BT)$

Ising Critical Point

$M(T) \sim (T_c-T)^\beta$

$\rho^{Ar}(T) = A(M(BT),T)$

Universality: Same Behavior up to Change in Coordinates

$A(M,T) = a_1 M + a_2 + a_3 T +$ *(other singular terms)*

Nonanalytic behavior at critical point (not parabolic top)

All power-law singularities (χ, c_v, ξ) are shared by magnets, liquid/gas

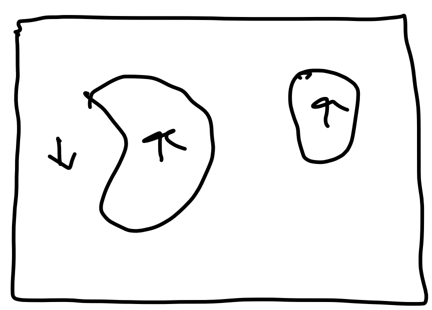
Fluctuations and the Ginzburg Criteria

Thus far we assumed the possibility of a Taylor expansion for the effective

$$H_{eff}(\phi) = E^0 + \alpha\phi + A\phi^2 + D\phi^3 + \cdots$$

This assumption alone led to the "mean field" $\alpha, \beta, \gamma, \delta$ which are a bit off from experiment. Interestingly, there is way to predict this failure within a generalization of Landau paradigm.

So far our order parameter was single # ϕ. Let's instead keep track of $\phi(\vec{x})$:

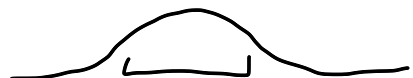


However, we don't want to keep track of <u>every</u> spin or we're back where we started!

Instead, we define a "coarse grained" ("blocked" / "smoothed") $\vec{\phi}(\vec{x})$. There are different concrete ways, for example

$$\phi(\underbrace{\vec{x}}_{\mathbb{R}^D} : \{\sigma\}) \equiv \sum_i \sigma_i \, w(\vec{x} - \vec{x}_i)$$

$$\int d^D w = 1 \qquad w =$$



width ~~"a"~~ Δ^{-1}

Note that by construction ϕ varies slowly over scale "a" (the "UV cutoff")

Now we obtain an Heff <u>functional</u>

$$e^{-\beta \, H_{eff}[\phi(x)]} \equiv \sum_\sigma e^{-\beta \mathcal{H}[\sigma]} \prod_x \delta(\phi(x) - \phi(\vec{x} : \{\sigma\}))$$

This would be terrible to compute concretely! But we can proceed phenomenologically by using "gradient" expansion,

$$\beta H_{eff}[\phi(x)] = \int d^D x \Big[h \cdot \phi(x) + \frac{a}{2} \phi^2(x) + \frac{u}{4} \phi^4(x) + \frac{K}{2} (\nabla \phi)^2 + S_0 \phi^2 (\nabla \phi)^2 + S_1 (\nabla \phi)^4 + S_2 \, \phi \, \nabla^4 \phi + \cdots$$

The exact relation between $h, a, u, \cdots$ and the coefficients in $\mathcal{H}$ are usually difficult to determine

Obviously we can keep going. But since ϕ varies slowly over ϕ, where a is correlation length. $\nabla\phi \sim \phi\Delta$. So leading in ϕ, $\cancel{a}$: Δ:

$$\mathcal{H}_{eff} = \int d^D x \left[\frac{a}{2}\phi^2(x) + \frac{K}{2}(\nabla\phi)^2 + \frac{u}{4}\phi^4(x) + \cdots \right]$$

Of course to compute Z, we have

$$Z = \int D[\phi]\, e^{-\beta \mathcal{H}_{eff}[\phi(x)]}$$

where $\int D[\phi] = \prod_x \int d\phi(x)$: a "path/field" integral. Many D.O.F, just like $\{\sigma\}$. However, near the C.P. ϕ is small, so we can <u>try</u> one further approximation: assume $\phi^4 \ll \phi^2$, and (for now) $a>0$. We'll need to test afterwards if this approximation is justified.

So, lets study stat-mech of

$$\mathcal{H}_{eff} = \frac{1}{2}\int \left[a\phi^2(x) + K(\nabla\phi)^2 \right]$$

"The Gaussian Point"

It can be solved by Fourier transformation + Gaussian integration:

$$\phi(q) \equiv \frac{1}{\sqrt{V}} \int e^{-iqx} \phi(x)\, d^D x$$

$$\nabla \longrightarrow -iq$$

$$\mathcal{H}_{eff} = \frac{1}{2} \sum_q \left[(a + \kappa q^2)\, \phi(q)\, \phi(-q) \right]$$

Since $\phi(q) = \phi^*(-q)$, we group together contribution from $q, -q$,

$$F = -\frac{1}{\beta} \ln(Z) = -\frac{1}{2\beta} \sum_q \ln\left(\int_{\mathbb{C}} d\phi_q\, e^{-\beta \epsilon_q |\phi_q|^2} \right)$$

where $\phi_q \in \mathbb{C}$, $\epsilon_q = a + \kappa q^2$

For $z = x + iy$, recall

$$\int dx\, dy\, e^{-\epsilon |z|^2} = \frac{\pi}{\epsilon}$$

So $$\boxed{F = -\frac{1}{2\beta} \sum_q \ln\left(\frac{\pi}{\epsilon_q \cdot \beta} \right)}$$

$$= -\frac{k_B T}{2} \sum_q \ln\left(\frac{\pi k_B T}{\epsilon_q} \right)$$

Correlation Functions

Let's use F to compute

$$C(r) = \langle \phi(r)\, \phi(0) \rangle,$$

which will quantify the distance over which spins align. By F.T,

$$C(r) = \frac{1}{V} \sum_{q,q'} e^{iqr} \langle \phi(q)\, \phi(q') \rangle$$

For $-q \neq q'$, $\langle \phi(q)\, \phi(q') \rangle = 0$.

by translation invariance. Otherwise,

$$\langle \phi(q)\, \phi(-q) \rangle = \partial_{\epsilon_q} F = \frac{1}{\beta} \frac{1}{\epsilon_q}$$

$$= \frac{1}{\beta} \frac{1}{a + \kappa q^2}$$

So

$$C(r) = \frac{1}{\beta \cdot V} \sum_q \frac{e^{iqr}}{a + \kappa q^2}$$

$$V \to \infty \quad \boxed{C(r) \simeq \frac{1}{\beta} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iqr}}{a + \kappa q^2}}$$

$$\beta C(r) \simeq \int \frac{d^D q}{(2\pi)^D} \frac{e^{iqr}}{a + \kappa q^2}$$

$$= \frac{1}{\kappa} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iqr}}{\xi^{-2} + q^2}, \quad \xi = \sqrt{\frac{\kappa}{a}}$$

"correlation length"

If $\xi = \infty$, this gives D-dim Coulomb:

$$\beta C(r) \propto \frac{1}{\kappa} \frac{1}{r^{D-2}}$$

Spins are power-law correlated out to infinite distance! Otherwise,

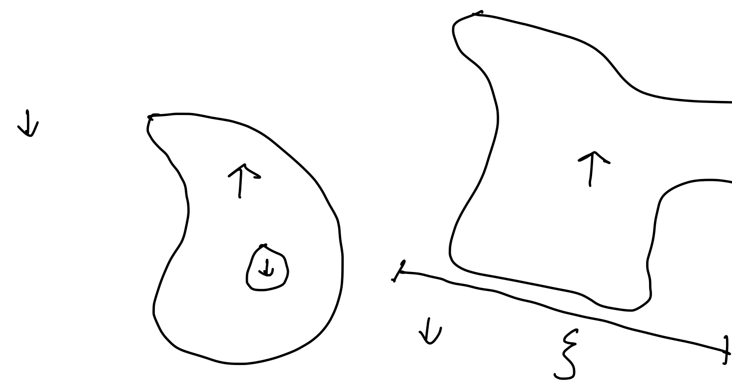
$$\beta C(r) = \frac{\xi^{2-D}}{\kappa} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \cdot \frac{r}{\xi}}}{1 + q^2}$$

This is D-dimensional "Yukawa potential". Since the integrand is analytic in the strip $|\mathrm{Im}(q)| \leq 1$, it's Fourier transform decays <u>exponentially</u> at large r. For

3D, $$\boxed{\beta C(r) = \frac{1}{4\pi\kappa} \frac{e^{-r/\xi}}{r}}$$

This is why ξ is "correlation length." In general, as $r \to \infty$ for fixed ξ,

$$c(r) \propto \frac{e^{-r/\xi}}{r^{(d-1)/2}} \qquad \text{(Arovas 7.205)}$$



So the picture is for $T > T_c$, system is organized into domains of size $\sim \xi$.

Since $\xi = \sqrt{\frac{\kappa}{a}}$, if we take $a = \alpha \frac{T - T_c}{T_c} \equiv \alpha \cdot t$

$$\boxed{\xi \propto t^{-\nu}, \quad \nu = \frac{1}{2}}$$

As $T \to T_c$, $\xi \to \infty$ and domain sizes are power-law distributed.

Now let's compute $C_V = -T\,\partial_T^2 F$
$= T \cdot \partial_T S$

$$F = -\frac{k_B T}{2} \sum_q \ln\left(\frac{\pi k_B T}{\epsilon_q}\right) \qquad -\partial_T F$$

At this point, we must remember

$\epsilon_q = a(T) + \kappa(T)\,q^2$ itself depends on T! Letting $a(T) = \alpha \cdot \underbrace{\frac{T-T_c}{T_c}}_{t} + \cdots$

$$S = -\partial_T F = \frac{k_B}{2}\sum_q \left\{ \ln\left(\frac{\pi k_B T}{\epsilon_q}\right) + 1 - \frac{\alpha/T_c}{\epsilon_q} \right\}$$

$$T\,\partial_T S\Big|_V = \frac{k_B T}{2}\sum_q \left\{ \frac{1}{T} - \frac{\alpha/T_c}{\epsilon_q} + \frac{(\alpha/T_c)^2}{\epsilon_q^2} \right\}$$

$$\frac{\epsilon_q^2 - T\frac{\alpha}{T_c}\epsilon_q + (\alpha/T_c)^2 \cdot T}{\epsilon_q^2\,T} = \frac{(\alpha - q^2\kappa)^2}{\epsilon_q^2\,T}$$

$$C = \frac{k_B}{2}\sum_q \frac{(\alpha - q^2\kappa)^2}{\underbrace{(\alpha t + q^2\kappa)^2}_{\epsilon_q^2}}$$

Now converting to $V \longrightarrow \infty$

$$\boxed{\frac{C}{V} = \frac{k_B}{2}\int \frac{d^D q}{(2\pi)^D}\,\frac{(\alpha - q^2\kappa^2)^2}{(\alpha t + q^2\kappa)^2}}$$

The interpretation of this is a bit subtle, because the integral may not converge for two distinct reasons: UV ($q \to \infty$) and IR ($q \to 0$)

In fact, for $q \to \infty$ the integrand goes as

$$\frac{C}{V} \ni \frac{k_B}{2} \int \frac{d^D q}{(2\pi)^D} \cdot 1 = \infty$$

$\phi(x)$ was a field that varied slowly over Λ^{-1}, which means $\phi(q) \to 0$ for $q/\Lambda \gg 1$. For example, with Gaussian coarse-graining $\phi(q) \sim e^{-q^2/\Lambda^2/2}$

But in our Gaussian model,

$$\langle \phi(q)\, \phi(-q) \rangle = \frac{1}{\beta} \frac{1}{a + k q^2}$$

which is much slower decay. In our H_{eff}, the fix really comes from higher-order $\phi \nabla^4 \phi$, etc., which kill large-q fluctuations. However, we can proceed phenomenologically coarse graining

by instead implementing a "hard" cutoff $|q| < \Lambda$ ~~$\frac{\pi}{a}$~~,

$$\frac{C}{V} = \frac{k_B}{2} \int_{|q|<\Lambda} \frac{d^D q}{(2\pi)^D} \frac{(\alpha - q^2 \kappa^2)^2}{(\alpha t + q^2 \kappa)^2}$$

Results which depend on Λ are sensitive to the details at scale-a; but we will find some results are <u>not</u> dependent on Λ.

If $t \to 0$, there is also an <u>IR</u> divergence coming from

$$\frac{C}{V} \ni \frac{k_B}{2} S_{D-1} \int_0^\Lambda \frac{dq\, q^{D-1}}{(2\pi)^D} \frac{\alpha^2}{(\alpha t + q^2 \kappa)^2} \sim \int dq\, q^{D-5}$$

$$= \frac{k_B \alpha^2}{2\kappa^2} \frac{S_{D-1}}{(2\pi)^D} \int_0^\Lambda dq \frac{q^{D-1}}{(\xi^{-2} + q^2)^2} ; \quad \xi \equiv (\alpha t)^{-1}$$

$$= \frac{k_B \alpha^2}{2\kappa^2} \frac{S_{D-1}}{(2\pi)^D} \xi^{4-D} \int_0^{\xi\Lambda} \frac{d\bar{q}\, \bar{q}^{D-1}}{(1 + \bar{q}^2)^2}$$

So $\frac{C/K_B}{(V/a^D)} \propto R_*^{-4} a^D \xi^{4-D} \int_0^{\xi\Lambda} \frac{d\bar{q}\, q^{D-1}}{(1+\bar{q}^2)^2}$

$R_* = \sqrt{\frac{\kappa}{\alpha}} = \xi(t=1)$, $\Lambda = 1/a$

Now let's estimate

$$\lim_{\xi\Lambda \to \infty} \int_0^{\xi\Lambda} \frac{d\bar{q}\, \bar{q}^{D-1}}{(1+\bar{q}^2)^2}$$

This can be done exactly, but the <u>scaling</u> comes from

$$\int_0^{\xi\Lambda} d\bar{q} \quad \bar{q}^{D-1-4} \sim \begin{cases} (\xi\Lambda)^{D-4}, & D>4 \\ \ln(\xi\Lambda), & D=4 \\ \text{const}, & D<4 \end{cases}$$

So $$\frac{C/K_B}{(V/a^D)} \propto \left(\frac{a}{R_*}\right)^4 \begin{cases} \text{const} & D>4 \\ \ln(\xi/a) & D=4 \\ (\xi/a)^{4-D} & D<4 \end{cases}$$

$$\frac{C/k_B}{(V/a^D)} \propto \left(\frac{a}{R_*}\right)^4 \begin{cases} \text{const} & D>4 \\ \ln(\xi/a) & D=4 \\ (\xi/a)^{4-D} & D<4 \end{cases}$$

So, for $D \leq 4$, there is a <u>singular</u> contribution to the heat capacity as $\frac{\xi}{R_*} \sim \frac{1}{\sqrt{t}} \to \infty$!

This means the fluctuations are important to the thermodynamics for $D \leq 4$. Indeed, MF Landau treatment predict $C \sim t^0$ ($\alpha = 0$) vs

$$C \sim t^{(4-D)/2} \quad (\alpha = (4-D)/2)$$

The fluctuation correction is $O(1)$ when

$$1 = \left(\frac{a}{R_*}\right)^4 \left(\frac{\xi}{a}\right)^{4-D} = \left(\frac{\xi}{R_*}\right)^{4-D} \cdot \left(\frac{a}{R_*}\right)^D$$

Using $\frac{\xi}{R_*} = \frac{1}{\sqrt{t}}$

$$\boxed{t_G = \left(\frac{a}{R_*}\right)^{\frac{2D}{4-D}}}$$

"Ginsburg Temp" $t_G = \left(\frac{a}{R_*}\right)^{\frac{2D}{4-D}}$

$$R_* = \xi(t=1)$$

For $t > t_G$, the fluctuating contribution to C is small, and MFT is ok; but for $t < t_G$, $D \leq 4$, the fluctuating contribution is important and MFT is corrected.