# Pressure of a quantum gas

For $\hat{H} = \sum \frac{p_i^2}{2m}$ in a $V = L^3$ box, the classical pressure is $\beta P = \frac{N}{V}$. Here we derive QM correction in the Grand ensemble (see Kardar 7.2 for Canonical)

The eigenstates are $|\vec{k}\rangle$ for

$$\vec{k} = \frac{2\pi}{L}(i, j, k), \quad \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

(non-relativistic) $\frac{\langle k|p^2|k\rangle}{2m}$

$$Q = \prod_{\vec{k}} \begin{cases} 1 + e^{-\beta(\epsilon_k - \mu)} & \text{Fer.} \\ (1 - e^{-\beta(\epsilon_k - \mu)})^{-1} & \text{Bos.} \end{cases}$$

$$\Omega = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} \sum_k \left(\mp \ln(1 \mp e^{-\beta(\epsilon_k - \mu)})\right)$$

Suppose gas is dilute (low-T limit), $\beta(\epsilon_k - \mu) \gg 1$ : $\ll 1$

$$\ln(1 + x) = -\sum_{m=1}^{\infty} (-1)^m \frac{x^m}{m}$$

$$\Omega_k = \mp\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(\pm)^m}{m} e^{-\beta m(\epsilon_k - \mu)}$$

Note $$\sum_k \simeq \left(\frac{L}{2\pi}\right)^D \int d^D k = V \cdot \int \frac{d^D k}{(2\pi)^D}$$

So $$\Omega = V \cdot \int \frac{d^D k}{(2\pi)^D} \Omega_k$$

$$\Omega = \frac{V}{\beta} \int \frac{d^D K}{(2\pi)^D} \left( \mp \sum_{m=1}^{\infty} \frac{(\pm)^m}{m} e^{-\beta m (\epsilon_K - \mu)} \right)$$

$$= \sum_{m=1}^{\infty} \Omega_m = \mp \frac{V}{\beta} \sum_m \frac{(\pm)^m}{m} e^{m\beta\mu} \left( \frac{1}{\lambda \sqrt{m}} \right)^D$$

$$= - \frac{V}{\beta \lambda^D} \sum_m \frac{(\pm)^{m-1}}{m} \frac{e^{\beta m \mu}}{\sqrt{m}^D}$$

$\lambda = \sqrt{\frac{\hbar^2 \pi}{k_B T m 2}}$

We then obtain P, N

$$P = -\frac{\partial \Omega}{\partial V} = \sum_m P_m = \frac{1}{\beta \lambda^D} \sum_m \frac{(\pm)^{m-1}}{m} \frac{e^{\beta m \mu}}{\sqrt{m}^D}$$

$$\frac{N}{V} = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} = \frac{1}{\lambda^D} \sum_m \frac{(\pm)^{m-1}}{\sqrt{m}^D} e^{\beta m \mu}$$

(c.f. Kardar 7.36)

Now recall virial expansion

$$\beta P = \frac{N}{V} \left( 1 + B_2(T) \frac{N}{V} + B_3(T) \left(\frac{N}{V}\right)^2 + \cdots \right)$$

Keeping to 2nd order in $e^{\beta\mu}$,

$$\left( e^{\beta\mu} \pm \frac{1}{2} \frac{e^{2\beta\mu}}{\sqrt{2}^D} + \cdots \right) = \left( e^{\beta\mu} \pm \frac{e^{2\beta\mu}}{\sqrt{2}^D} + \cdots \right) \times \left( 1 + B_2(T) \frac{e^{\beta\mu}}{\lambda^D} + \cdots \right)$$

2nd Order:

$$\pm \frac{1}{2} \frac{e^{2\beta\mu}}{\sqrt{2}^D} = \pm \frac{e^{2\beta\mu}}{\sqrt{2}^D} + B_2(T)\, e^{2\beta\mu} / \lambda^D$$

$$\pm \frac{1}{2} \frac{e^{2\beta\mu}}{\sqrt{2}^D} = \pm \frac{e^{2\beta\mu}}{\sqrt{2}^D} + B_2(T)\, e^{2\beta\mu}/\lambda^D$$

$$B_2(T) = \mp \frac{\lambda^D}{2^{(D+2)/2}}$$

2nd Virial Coefficient of quantum gas

For Bosons, negative correction to pressure; for fermions, positive.

Classically, for pairwise interactions

$$B_2 = -\frac{1}{2} \int d^3q \left( e^{-\beta V(q)} - 1 \right)$$

In Kardar 7.2, this is shown to be consistent with quantum result if we take

$$\beta V(r) = \mp e^{-2\pi r^2/\lambda^2}$$

So bosons/fermions behave as if there is short-range $(r \sim \lambda)$ attractive/repulsive interaction. Of course really, there is no interaction: it is quantum statistics!

Important when $\lambda^D \cdot \frac{N}{V} \approx 1$

"Quantum Degeneracy"

We'll study the quantum degenerate limits next.