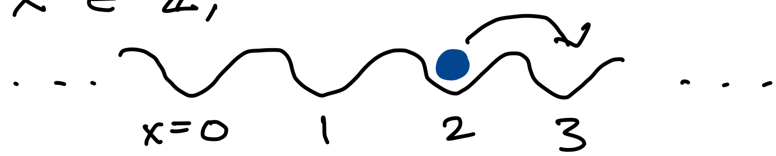
Bosons

Consider particle with Hilbert space $|x\rangle$. Usually we think $x \in \mathbb{R}^D$. However, in AMO/CM, we might also consider $x \in \mathbb{Z}^D$,



It won't make a difference in the following except for $\int dx$ vs $\sum_x$:

$$\mathbb{1} = \int d^D x \; |x\rangle\langle x| \quad \text{vs} \quad \mathbb{1} = \sum_x |x\rangle\langle x|$$

and $\langle x|y\rangle = \delta(x-y)$ vs $\langle x|y\rangle = \delta_{x,y}$

In these notes, I'll use $\sum_x$ notation.

The wavefunction is $\psi(x) = \langle x|\psi\rangle$

For <u>two</u> particles, the Hilbert space is spanned by

$$|x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$$

$$\psi(x_1, x_2) = \langle x_1, x_2|\psi\rangle$$

And we can continue: $\psi(x_1, x_2, \cdots x_N)$

We now arrive at a deep physical fact. If the particles are the same species (i.e, all W-bosons or Ru), $|x_1, x_2\rangle$ is in fact same quantum state as $|x_2, x_1\rangle$.

For bosons, $|x_1, \ldots, x_i, \cdots x_j, \cdots x_N\rangle = |x_1, \ldots x_j, \cdots x_i, \cdots, x_N\rangle$

"Exchange symmetry"

What does this mean?

First, for any physical observable,

$$\langle x_1, x_2 | \mathcal{O} | x_1, x_2\rangle = \langle x_2, x_1 | \mathcal{O} | x_2, x_1\rangle$$

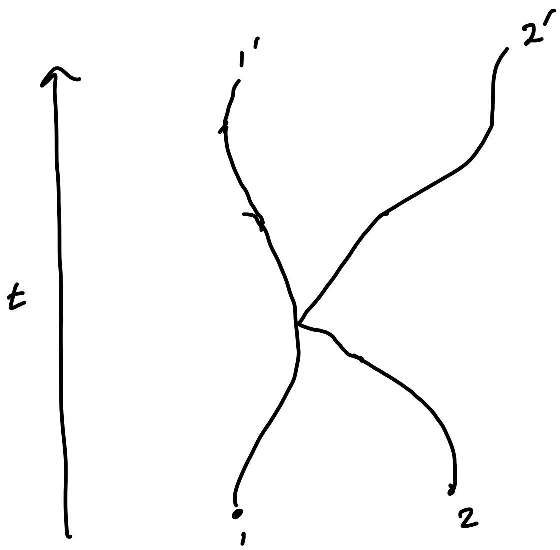
No way to measure "which" boson at x; only that a boson is at x.

This constrains the $\mathcal{O}$ we'll deal with. For example, the avg. num of particles at site x is

$$n(x) = \sum_{x_2} \psi^*(x_1 = x, x_2)\, \psi(x_1 = x, x_2)$$

$$+ \sum_{x_1} \psi^*(x_1, x_2 = x)\, \psi(x_1, x_2 = x)$$

__Second__, it allows for constructive interference. In QM, the amplitude to evolve from $12 \to 1'2'$ is a __sum over all paths__ connecting initial / final state. For bosons, these paths include trajectories which involve exchange:



Path 1

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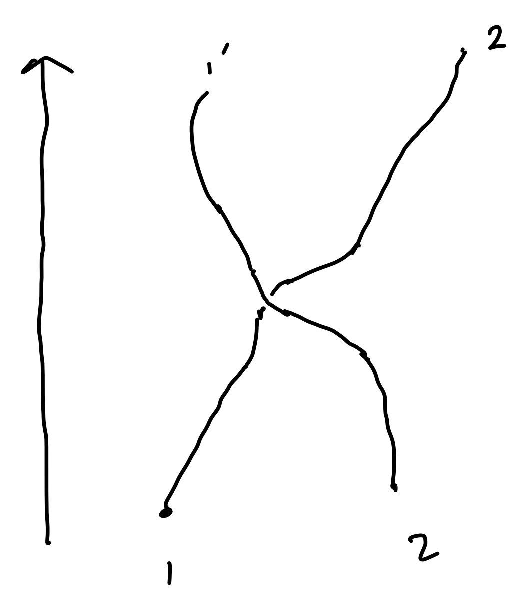

since $|x_1, x_2\rangle = |x_2, x_1\rangle$!

This has measurable consequences for analogs of 2-slit experiment, e.g. "Hanbury - Brown - Twiss"

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\int dx\, f(x)$$

First + Second Quantized W.F.s

In the <u>first-quantized</u> language, we consider $\psi(x_1, x_2, \cdots)$ and demand $\psi(\cdots, x_i, \cdots, x_j \cdots) = \psi(\cdots x_j, \cdots, x_i \cdots)$

$$|\psi\rangle = \sum_{x_i} \psi(\{x_i\}) \, |\{x_i\}\rangle$$

If P_{ij} swaps $i \leftrightarrow j$, we can think of $P_{ij}\psi = \psi$ as a symmetry. Is this demand consistent with time-evolution? <u>Yes</u>, because physically admissable H also preserve P_{ij}:

$$H = \sum_i \frac{p_i^2}{2m} + V(x_i) + \frac{1}{2} \sum_{i \neq j} U(x_i - x_j) + \cdots$$

$$P_{ij}^{-1} H P_{ij} = H$$

So $i\partial_t \psi = H\psi \Rightarrow P_{ij}\psi(t) = \psi(t)$ for all t if $P_{ij}\psi(0) = \psi(0)$.

<u>Forbids</u> terms like $V_1(x_1) + V_2(x_2)$ for $V_1 \neq V_2$ since this distinguishes $1/2$.

2nd Quantized Language

Because of symmetry, we see w.f. can always be expanded in terms of

$$|x_1, x_2\rangle_+ \equiv \begin{cases} \frac{1}{\sqrt{2}}\left(|x_1, x_2\rangle + |x_2, x_1\rangle\right) & \text{if } x_1 \neq x_2 \\ |x_1, x_2\rangle & \text{if } x_1 = x_2 \end{cases}$$

More generally, $|\{x_i\}\rangle_+ \equiv \frac{1}{\sqrt{\#\text{Perm}}} \sum_{\text{Perm}} |\text{Perm}\{x_i\}\rangle$

By definition, $P_{ij} |\{x\}\rangle_+ = |\{x\}\rangle_+$, e.g. $|1,3\rangle_+ = |3,1\rangle_+$. Since order doesn't matter, this suggests different way to label: we simply count how-many particles "n_x" occupy state x!

For $N=2$ particles in $x = 1,2,3$ orb!

$$|x_1 = 1, x_2 = 3\rangle_+ = \frac{1}{\sqrt{2}}\left(|1,3\rangle + |3,1\rangle\right)$$

$$\hookrightarrow \quad = |n_1 = 1, n_2 = 0, n_3 = 1\rangle = |101\rangle$$

$$|1,2\rangle_+ = |n_1 = 1, n_2 = 1, n_3 = 0\rangle = |110\rangle$$

$$|1,1\rangle_+ = |n_1 = 2, n_2 = 0, n_3 = 0\rangle = |200\rangle$$

The string $|\{n_x\}\rangle$ is called the "occupation basis."

$$N = \sum_x n_x = \text{total number}$$

Note that $\{n_x\}$ depends on our choice of single-particle basis. For example, rather than basis $|x\rangle$, could use

$$|x\rangle \longrightarrow |k\rangle$$

$$|x_1, x_2, \cdots\rangle \longrightarrow |k_1, k_2 \cdots\rangle$$

$$|\{n_x\}\rangle \longrightarrow |\{n_k\}\rangle$$

Using $\langle x|k\rangle = e^{ikx}/\sqrt{L}$, one can work out

$$\langle x_1, x_2, \cdots | k_1, k_2, \cdots\rangle = \frac{1}{\sqrt{L^N}} e^{i\sum_i x_i \cdot k_i}$$

However, computing $\langle\{n_x\}|\{n_k\}\rangle$ is a bit tricky! Called the "Boson sampling problem": #P-hard.

You'll practice on H.W.

Suppose that $\hat{H}$ is non-interacting,

$$\hat{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(x_i)$$

Let $\left(\frac{p^2}{2m} + V(x)\right)|\alpha\rangle = E_\alpha|\alpha\rangle$ be
$\alpha = 0, 1, 2, \cdots$

<u>single-particle</u> eigenstates; $|x\rangle \Rightarrow |\alpha\rangle$

Many-body spanned by $|\alpha_1, \alpha_2, \cdots, \alpha_N\rangle$ (first-quant.) or, in occupation basis,

$$|\{n_\alpha\}\rangle$$

In this basis

$$\hat{H} = \sum_\alpha \hat{n}_\alpha \cdot E_\alpha = \sum_\alpha a_\alpha^\dagger a_\alpha E_\alpha$$

where $\hat{n}_\alpha | n_0, n_1, n_2, \cdots\rangle = n_\alpha | n_0, n_1, n_2, \cdots\rangle$

We see that $\hat{H}$ is formally equivalent to decoupled Harmonic oscillators, with

$\hbar\omega_\alpha = E_\alpha$ and $n_\alpha = \#$ of quanta in oscillator "α"

This motivates bosonic raising/lowering operators:

$$\hat{n}_\alpha = a^+_\alpha a_\alpha$$

$$[a_\alpha, a^+_\beta] = \delta_{\alpha\beta}$$

why operator defined by commutator relations? easy

$$a^+_1 |n_1, n_2, n_3, \cdots\rangle = \sqrt{n_1+1}\, |n_1+1, n_2, n_3, \cdots\rangle$$

$$a_1 |n_1, \cdots\rangle = \sqrt{n_1}\, |n_1 - 1, \cdots\rangle$$

So a^+_α "creates" a boson in state α

a_α "destroys" a boson in state α

Note that under single particle unitary transformation $|\alpha\rangle = \sum_x U_{\alpha,x} |x\rangle$

$$a^+_\alpha = \sum_x U_{\alpha,x} a^+_x$$

$$[a_\alpha, a^+_\beta] = \sum_{x,y} U^*_{\alpha,x} \underbrace{[a_x, a^+_y]}_{\delta_{x,y}} U_{\beta,y}$$

$$= \sum_x U_{\beta,x} U^*_{\alpha,x} = \delta_{\alpha\beta}$$

A <u>non-interacting</u> $\hat{H} = \sum_{x,y} a^+_x H_{x,y} a_y$

$$= \sum_\alpha E_\alpha a^+_\alpha a_\alpha$$

The stat-mech of non-interacting $H = \sum \epsilon_\alpha \hat{n}_\alpha$ is simple in Grand ensemble. Since $[\hat{n}_\alpha, \hat{n}_\beta] = 0$,

$$e^{-\beta\left(\sum \epsilon_\alpha \hat{n}_\alpha - \mu N\right)} = e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$Z = \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$= \prod_\alpha \left(\sum_n e^{-\beta n (\epsilon_\alpha - \mu)} \right)$$

$$= \prod_\alpha \frac{1}{1 - e^{-\beta(\epsilon_\alpha - \mu)}}$$

which gives Bose-distribution

$$\langle n_\alpha \rangle = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}$$

Fermions

For fermions, $P_{ij}\,\psi = -\psi$

$$\psi(x_1, x_2) = -\psi(x_2, x_1)$$

So we define

$$|x_1, x_2, \cdots\rangle_- \equiv \frac{1}{\sqrt{\#\text{Perm}}} \sum_{\text{Perm}} (-1)^P \,|\text{Perm}\{x\}\rangle$$

e.g. $|1,2\rangle_- = \frac{1}{\sqrt{2}}\left(|1,2\rangle - |2,1\rangle\right)$

Note $|\{x_i\}\rangle_- = 0$ if any $x_i = x_j$

So restrict to $x_i \neq x_j$: "Pauli exc."

Note $|x_1, x_2, x_3 \cdots\rangle_- = -|x_2, x_1, x_3 \cdots\rangle_-$

So we can restrict to representatives $x_1 < x_2 < x_3 < \cdots$

Then occupation basis is

$$|n_1 = 1, n_2 = 0, n_3 = 1\rangle = |1, 3\rangle_-$$

etc.

But now $n_x = 0$ or 1 because of of exclusion.

$$\mathcal{H} = \text{span}\left(|110010\cdots\rangle, |11110\rangle \cdots\right)$$

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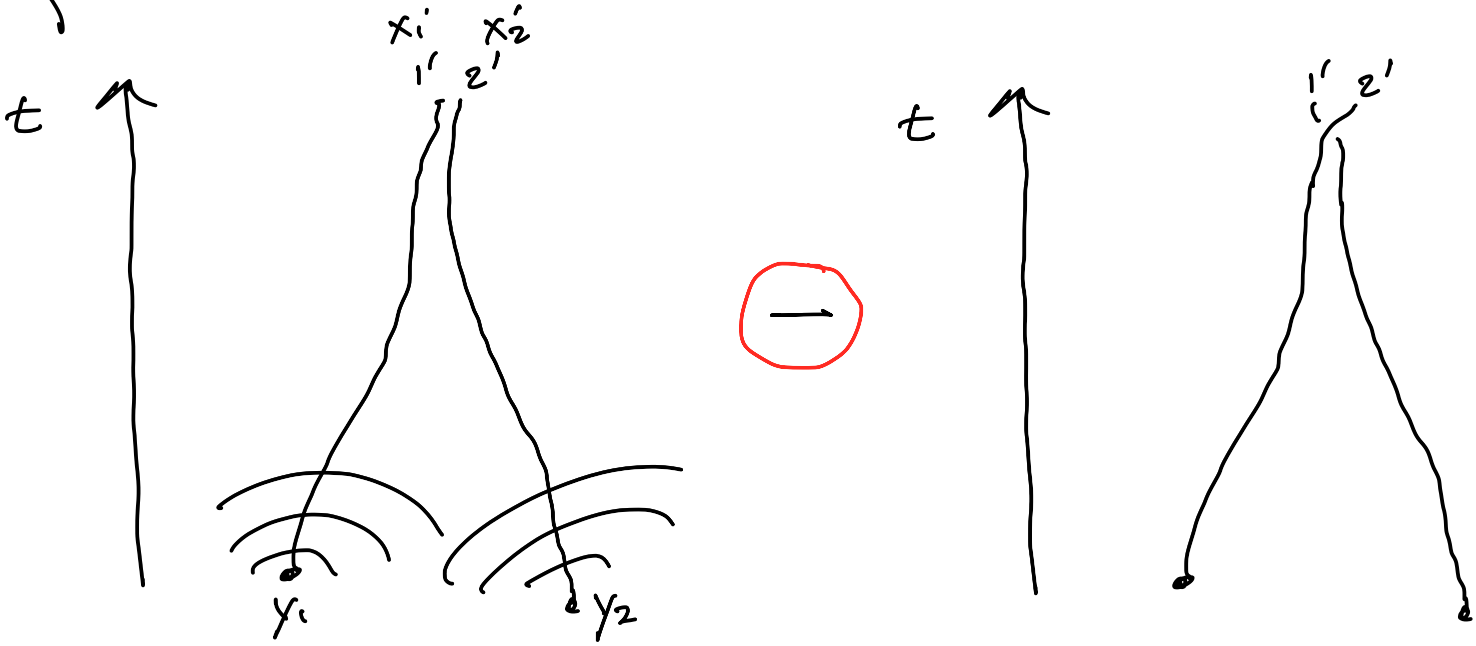
Exclusion can be understood as <u>destructive interference</u> phenomena:



Path 1 Path 2

$w(x) =$

[See: R. Shankar's QM]

$$\lim_{1' \to 2'} \left(e^{iS_1/\hbar} - e^{iS_2/\hbar} \right) \to 0$$

$$\lim_{1' \to 2'} S_1 = S_2 \qquad S = \int \mathcal{L}(x)\, dt$$

Amplitude vanishes for fermions to end up at same place!

$$\psi(x_1, x_2, t=0) = \frac{1}{\sqrt{2}} \left(w(x_1 - y_1)\, w(x_2 - y_2) - \leftrightarrow \right)$$

$\downarrow$ Schrodinger.

$$\psi(x_1, x_2, t)$$

$$P_{12 \to 1'2'} \quad |\psi(x_1', x_2', t)|^2$$

For non-interacting fermions, we thus have $H = \sum_\alpha \epsilon_\alpha \hat{n}_\alpha \quad \hat{n}_\alpha = 0/1$

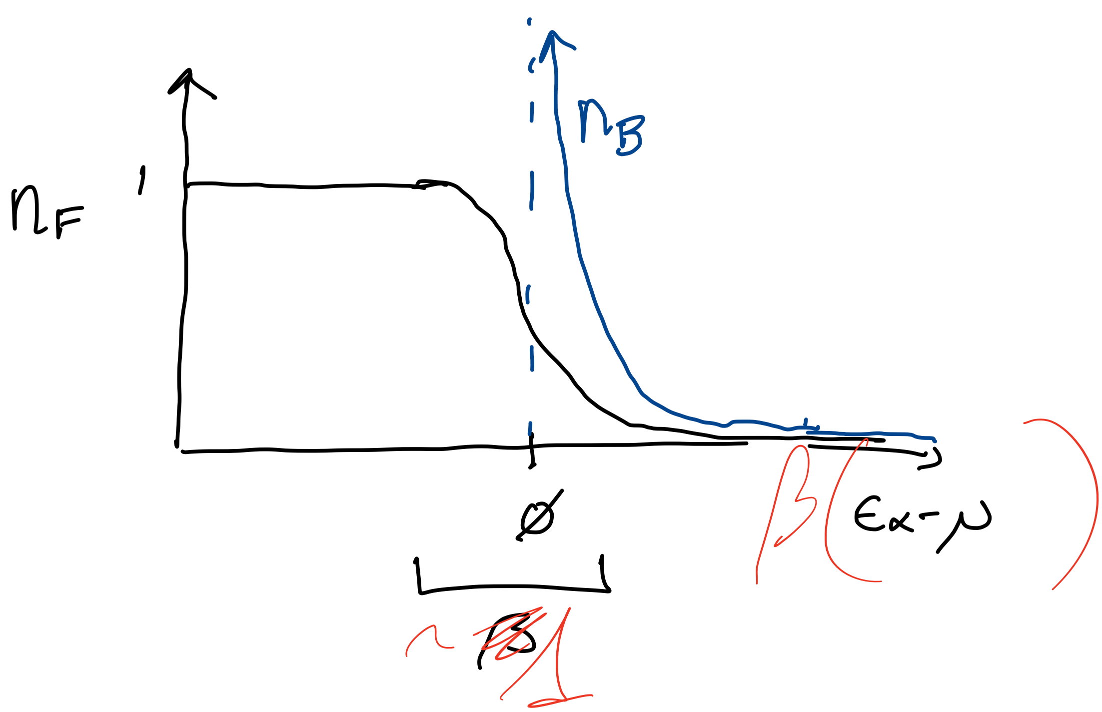
$$Z = \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$= \prod_\alpha \left(\sum_{n=0}^{1} e^{-\beta(\epsilon_\alpha - \mu) n} \right)$$

$$= \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)} \right)$$

$$\langle n_\alpha \rangle = \frac{e^{-\beta(\epsilon_\alpha - \mu)}}{1 + e^{-\beta(\epsilon_\alpha - \mu)}} = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}$$

This motivates $\langle n \rangle_{B/F} = \dfrac{1}{e^{\beta(\epsilon_\alpha - \mu)} \mp 1}$



Fermionic raising/lowering operators

Similar to bosons, we define ops

c^+_α, c_α, $n_\alpha = c^+_\alpha c_\alpha$.

Let us start with single orbital $\alpha = 1$:

⚠ Pre-factors!

$$c^+ |n\rangle = |n+1\rangle \qquad c^+|1\rangle = 0$$

$$c\, |n\rangle = |n-1\rangle \qquad c|0\rangle = 0$$

But because $n_\alpha = 0, 1$

$$c_\alpha^2 = (c^+_\alpha)^2 = 0, \qquad c^+_\alpha c_\alpha = n_\alpha$$

For a single orbital,

$$c = \begin{array}{c|c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} & \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \end{array} \quad c^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

But this implies $c\,c^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$c^+ c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = n$$

So $$c^+ c + c\,c^+ = \mathbb{1} = \{c^+, c\}$$

$$\{A, B\} \equiv AB + BA$$

"anti-commutation relation"

$$c^{+2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \checkmark$$

Ladder op's operating on different orbitals, c_1 and c_2
~~But what about c_1 + c_2~~ ?

Do we have $\{c_1^+, c_2^+\} = 0$ or $\underbrace{[c_1^+, c_2^+] = 0}_{\text{wrong!}}$??

~~Now we need to be careful because $|x_1, x_2\rangle_- = -|x_2, x_1\rangle_-$!!~~

Let us <u>define</u>

$$c_x^+ |x_1, \cdots, x_N\rangle_- = \overbrace{|x_1, \cdots, x_N, x_{N+1} = x\rangle_-}^{=0 \text{ if } x_i = x \text{ for any } i \in \{1, N\} \ \triangle!}$$

Now

$$c_y^+ c_x^+ |x_1, \cdots, x_N\rangle = c_y^+ |x_1, \cdots, x_N, x_{N+1} = x\rangle_-$$
$$= |x_1, \cdots, x_N, x_{N+1} = x, x_{N+2} = y\rangle_-$$

But similarly

$$c_x^+ c_y^+ |x_1, \cdots, x_N\rangle = |x_1, \cdots, x_N, x_{N+1} = y, x_{N+2} = x\rangle_-$$

This implies $c_x^+ c_y^+ = - c_y^+ c_x^+$

$$\hookrightarrow \boxed{\begin{aligned} \{c_x^+, c_y^+\} &= 0 \\ \{c_x, c_y\} &= 0 \\ \{c_x^+, c_y\} &= \delta_{x,y} \end{aligned}}$$

$\triangle!$ Bosons → commutators || Fermions → anticomm's

Because $|\{n_\alpha\}\rangle \equiv |x_1, x_2, \cdots\rangle_-$ for $x_1 < x_2 < \cdots$,

$$|x_1, x_2, \cdots\rangle_- = c^+_{x_N} \cdots c^+_{x_2} c^+_{x_1} |\rangle_-$$

<u>if</u> $x_1 < x_2, \cdots$.

But this leads to "−" signs if you create in a different order! In particular

$$c^+_\alpha |n_1, \cdots, n_\alpha, \cdots\rangle = \eta\, |n_1, \cdots, n_\alpha + 1, \cdots\rangle$$

where $\eta = \underbrace{(-1)^{\sum_{\beta > \alpha} n_\beta}}_{\text{"Jordan Wigner String"}}$

So even though the Hilbert space of fermions <u>looks</u> like $S = 1/2$ spins,

$$|\{n_\alpha\}\rangle = |1001101\cdots\rangle \sim |\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\cdots\rangle$$

fermion $\hat{H}$ are built from c^+_α, c_α which behave different from S^+_α, S^-_α!